MATH 119 Chapters 5-6

0.1 Theorem 0

Let c be in the domain $\mathcal{D}$ of f. If f is differentiable at c, then f is continuous at c.

0.2 Definition 1: Increasing and Decreasing Functions

i) A function f is increasing in an interval I contained in the domain $\mathcal{D}$ of f if

$$f(x_1) < f(x_2), \qquad \text{whenever } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2$$

ii) A function f is decreasing in an interval I contained in the domain $\mathcal{D}$ of f if

$$f(x_1) > f(x_2), \qquad \text{whenever } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2$$

0.3 Theorem 1

Let $\mathcal{D}$ be the domain of the function f and I an interval contained in $\mathcal{D}$.

i) If $f'(x) > 0$ for all x in I, then f is increasing in I.

ii) If $f'(x) < 0$ for all x in I, then f is decreasing in I.

0.4 Definition 2: Critical Numbers and Critical Points

i) A critical number for a function f is a number c in the domain $\mathcal{D}$ of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

ii) A critical point for a function f is a pair $(c, f(c))$ such that c is a critical number.

0.5 Definition 3: Relative Maximum and Relative Minimum for f

Let c be in the domain $\mathcal{D}$ of f, then

i) $f(c)$ is a relative or local maximum for f if there exists an open interval (a, b) contained in $\mathcal{D}$ and containing c such that $f(x) \leq f(c)$ for all x in (a, b).

ii) $f(c)$ is a relative or local minimum for f if there exists an open interval (a, b) contained in $\mathcal{D}$ and containing c such that $f(x) \geq f(c)$ for all x in (a, b).

0.6 Theorem 2: Necessary Condition for Relative Extrema

Let c be in the domain of f. If f has a relative maximum or minimum (relative extremum) at c, then c is a critical number for f. It means $f'(c) = 0$ or $f'(c)$ does not exist.

0.7 Theorem 3: First Derivative Test

Let c be the only critical number for f in (a, b). Consider a function f differentiable on (a, b) except possibly at c.

i) If $f'(x)$ is positive in the interval (a, c) and negative in the interval (c, b), then $f(c)$ is a relative or local maximum of f in (a, b).

ii) If $f'(x)$ is negative in the interval (a, c) and positive in the interval (c, b), then $f(c)$ is a relative or local minimum of f in (a, b).

ii) If $f'(x)$ has the same sign in the intervals (a, c) and (c, b), then $f(c)$ is neither a local maximum nor a local minimum.

0.8 Definition 4: Higher Derivatives

Consider a function f defined in a domain $\mathcal{D}$ which is differentiable in $\mathcal{D}'$ contained in $\mathcal{D}$. If the derivative function f' is also differentiable in $\mathcal{D}''$ contained in $\mathcal{D}'$, then the new function $(f')'$ will be called the second derivative of f and will be denoted as

$$f'', \quad \text{or} \quad \frac{d^2y}{dx^2}, \quad \text{or} \quad \frac{d^2f}{dx^2}.$$

For a in $\mathcal{D}''$, $f''(a)$ is called the second derivative of f at a. In similar way, derivatives of third, fourth and higher orders are defined.

0.9 Definition 5: Concavity

Consider a differentiable function f on an open interval (a, b).

1. A function f is concave upward on an interval (a, b) if the graph of f lies above its tangent line at each point of (a, b), except at $(a, f(a)$ itself.
2. A function f is concave downward on an interval (a, b) if the graph of f lies below its tangent line at each point of (a, b), except at $(a, f(a)$ itself.

0.10 Definition 6: Inflection Point

Let f be continuous at c. If the graph of f is concave upward on one side of c and concave downward on the other side, then the point $(c, f(c))$ is called an inflection point of f.

0.11 Theorem 4: Test for Concavity

Consider a function f defined on (a, b) with derivatives f' and f'' also defined on (a, b).

1. If $f''(x) > 0$ for all x in (a, b), then f is concave upward on (a, b).
2. If $f''(x) < 0$ for all x in (a, b), then f is concave downward on (a, b).

0.12 Theorem 5: Necessary condition for Existence of an Inflection Point

If $(c, f(c))$ is a point of inflection of f, then $f''(c) = 0$ or $f''(c)$ does not exist.

0.13 Theorem 6: Second Derivative Test for Relative Extrema

Consider a function f whose first derivative f' exists on interval (a, b) containing c, $f'(c) = 0$, and $f''(c)$ exists.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

0.14 Definition 6: Absolute Maximum or Minimum

Consider a function f defined on an interval I (open or closed) and the point c in I.

1. If $f(x) \leq f(c)$ for all x in I, then $f(c)$ is the absolute maximum of f on I.
2. If $f(x) \geq f(c)$ for all x in I, then $f(c)$ is the absolute minimum of f on I.

0.15 Theorem 7: Extreme Value Theorem

Any continuous function f on a closed interval $I = [a, b]$ has an absolute maximum and also an absolute minimum on I.

0.16 Finding the absolute maximum and absolute minimum values of f on a closed interval [a,b]

Let $[a, b]$ be a closed interval contained in the domain $\mathcal{D}$ of f and assume f has a maximum value and a minimum value on $[a, b]$ (if f were continuous that would be the case). To locate the points where the maximum and minimum value are reached, three kinds of points must be considered:

i) The critical numbers of f in (a, b). It means points in (a, b) where f is differentiable or not.

ii) The endpoints a and b.

Then, f should be evaluated at these points (if they are not too many). The biggest of these values will be the absolute maximum value of f, and the smallest will be the absolute minimum value on $[a, b]$.